

# ON FREE PROFINITE SUBGROUPS OF FREE PROFINITE MONOIDS

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**ABSTRACT.** We answer a question of Margolis from 1997 by establishing that the maximal subgroup of the minimal ideal of a finitely generated free profinite monoid is a free profinite group. More generally if  $\mathbf{H}$  is variety of finite groups closed under extension and containing  $\mathbb{Z}/p\mathbb{Z}$  for infinitely many primes  $p$ , the corresponding result holds for free  $\text{pro-}\overline{\mathbf{H}}$  monoids.

## 1. INTRODUCTION

Margolis asked in 1997 whether the maximal subgroup of the minimal ideal of a finitely generated free profinite monoid is a free profinite group; this question first appeared in print (to the best of our knowledge) in our paper with Rhodes [10]. The question was prompted by the discovery of free profinite subgroups by Almeida and Volkov [4], who subsequently characterized those free profinite subgroups which are retracts [5]. Recently, Almeida has shown that not all maximal subgroups of finitely generated free profinite monoids are free profinite groups [3], although he has provided a large class of examples that are free profinite. His fascinating technique involves a correspondence between symbolic dynamical systems in  $X^\omega$  and certain  $\mathcal{J}$ -classes of the free profinite monoid  $\widehat{X}^*$ . In particular, his methods apply best to *maximal* infinite  $\mathcal{J}$ -classes, which correspond to *minimal* dynamical systems. The minimal ideal corresponds to the full shift  $X^\omega$  and so Almeida's approach does not yet apply to studying this maximal subgroup.

The author and Rhodes recently established that closed subgroups of free profinite monoids are projective profinite groups [11]. This answered a question raised by several people including Almeida, Margolis, Lubotzky and the author [10]. Projectivity is a necessary, but far from sufficient condition, for freeness [13]. In this paper we answer Margolis's question in the affirmative. We also prove the analogous result relative to certain varieties of finite groups. Recall that if  $\mathbf{H}$  is a variety of finite groups, that is a class of finite groups closed under taking direct products, subgroups and

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quotient groups, then the class  $\overline{\mathbf{H}}$  of monoids whose subgroups belong to  $\mathbf{H}$  is a variety of finite monoids. Our main result is then:

**Theorem 1.** *Let  $\mathbf{H}$  be a variety of finite groups closed under extension, which contains  $\mathbb{Z}/p\mathbb{Z}$  for infinitely many primes  $p$ . Then the maximal subgroup of the minimal ideal of a finitely generated (but not procyclic) free pro- $\overline{\mathbf{H}}$  monoid is a free pro- $\mathbf{H}$  group of countable rank.*

If  $\mathbf{V}$  is a variety of finite monoids, then  $\widehat{F}_{\mathbf{V}}(X)$  denotes the free pro- $\mathbf{V}$  monoid generated by  $X$ . The natural projection  $\pi : \widehat{F}_{\overline{\mathbf{H}}}(X) \rightarrow \widehat{F}_{\mathbf{H}}(X)$  restricts to an epimorphism on the maximal subgroup  $G$  of the minimal ideal of  $\widehat{F}_{\overline{\mathbf{H}}}(X)$  [5, 10]. Our second result describes the kernel of the epimorphism  $G \twoheadrightarrow \widehat{F}_{\mathbf{H}}(X)$ .

**Theorem 2.** *Let  $\mathbf{H}$  be a variety of finite groups closed under extension, containing  $\mathbb{Z}/p\mathbb{Z}$  for infinitely many primes  $p$ , and let  $X$  be a finite set of cardinality at least two. Let  $\varphi : G \twoheadrightarrow \widehat{F}_{\mathbf{H}}(X)$  be the canonical epimorphism, where  $G$  is the maximal subgroup of the minimal ideal of  $\widehat{F}_{\overline{\mathbf{H}}}(X)$ . Then  $\ker \varphi$  is a free pro- $\mathbf{H}$  group of countable rank.*

It seems likely that our results hold for any non-trivial extension-closed variety of finite groups. The hypothesis on primes is entirely of a technical nature and should not really be essential. For example, since projective pro- $p$  groups are free pro- $p$  [13], Theorems 1 and 2 are valid for  $\mathbf{H}$  the variety of finite  $p$ -groups. We further propose the following conjecture.

**Conjecture 3.** *Under the hypotheses of Theorem 1 the maximal subgroup of the closed subsemigroup generated by the idempotents of the minimal ideal of a finitely generated (non-procyclic) free pro- $\overline{\mathbf{H}}$  monoid is a free pro- $\mathbf{H}$  group of countable rank.*

In fact, we suspect a slight variation of the construction used to prove Theorem 1 already suffices to prove the conjecture, the remaining issues being purely technical. The proof of Theorem 1 relies on a criterion for freeness, due to Iwasawa [8], and extensive usage of wreath products. In spirit the proof draws from the following sources: our previous work with Rhodes [11], the synthesis theorem [1] and the classical construction embedding any countable group as a maximal subgroup of a two-generated monoid consisting of a cyclic group of units and a completely simple minimal ideal.

## 2. MINIMAL IDEALS

In this section we collect a number of standard facts concerning minimal ideals in finite and profinite semigroups, which can be found, for instance, in [6, 9, 12]. If  $S$  is a semigroup, then  $E(S)$  denotes the set of idempotents of  $S$ . For an idempotent  $f \in E(S)$ , the group of units  $G_f$  of the monoid  $fSf$  is called the maximal subgroup of  $S$  at  $f$ .

The first fact is that every profinite monoid  $M$  has a unique minimal ideal  $I$ . It is necessarily closed and if  $x \in I$ , then  $I = MxM$ . Since

every compact semigroup contains an idempotent, it follows that  $I$  contains an idempotent  $e$ . Since compact semigroups are stable [12], Green-Rees structure theory [6, 9, 12] implies that the maximal subgroup  $G_e$  is  $eIe$  and furthermore is a closed subgroup (and hence a profinite group), which is independent of the choice of  $e$  up to isomorphism.

**Proposition 4.** *Let  $\varphi : S \twoheadrightarrow T$  be a continuous onto homomorphism of profinite monoids. Let  $I$  be the minimal ideal of  $S$  and  $J$  be the minimal ideal of  $T$ . Then  $\varphi(I) = J$  and moreover, if  $e \in E(I)$ , then  $\varphi(G_e)$  is the maximal subgroup of  $J$  at  $\varphi(e)$ .*

*Proof.* Clearly  $\varphi^{-1}(J)$  is an ideal of  $S$  so  $I \subseteq \varphi^{-1}(J)$ , i.e.  $\varphi(I) \subseteq J$ . On the other hand  $\varphi(I)$  is an ideal of  $T$  since  $\varphi$  is onto. Thus  $\varphi(I) = J$  by minimality. Now  $\varphi(G_e) = \varphi(eIe) = \varphi(e)\varphi(I)\varphi(e) = \varphi(e)J\varphi(e) = G_{\varphi(e)}$ , completing the proof.  $\square$

In particular, every profinite group image of a profinite monoid  $M$  is an image of the maximal subgroup of its minimal ideal.

The minimal ideal  $I$  of a finite monoid  $M$  is a simple semigroup, and hence isomorphic to a Rees matrix semigroup  $\mathcal{M}(G, A, B, C)$  where  $C : B \times A \rightarrow G$  is the sandwich matrix [6, 9, 12]. Let  $a_0 \in A$  and  $b_0 \in B$ . Then without loss of generality we may assume that each entry of row  $b_0$  and of column  $a_0$  is the identity of  $G$  [9, 12]. We can identify  $G$  with the  $\mathcal{H}$ -class  $a_0 \times G \times b_0$ .

Recall that  $B$  can be identified with the  $\mathcal{L}$ -classes of  $I$ . There is a natural action of  $M$  on the right of  $B$  since  $\mathcal{L}$  is a right congruence. Let  $(B, \text{RLM}_I(M))$  be the associated faithful transformation semigroup. Notice that each element of  $I$  acts on  $B$  as a constant map and that all constant maps on  $B$  arise from elements of  $I$ . The Schützenberger representation gives a wreath product representation  $M \rightarrow G \wr (B, \text{RLM}_I(M))$  [6, 9, 12]. An element  $s = (a, g, b) \in I$  is sent to the element  $(f_s, \bar{b})$  where  $b'f_s = b'Cag$  and  $\bar{b}$  is the constant map to  $b$ . In particular, if  $s = (a_0, g, b_0)$  is an element of our maximal subgroup, then  $b'f_s = g$  all  $b' \in B$ . Consequently, the Schützenberger representation is faithful on the maximal subgroup  $G$ .

We recall that the finite simple semigroups form a variety of finite semigroups denoted **CS**. It is well known [2] that **CS** = **G** \* **RZ**, that is, it consists precisely of divisors of wreath products of finite groups and right zero semigroups. In fact, we shall need the following more explicit lemma.

**Lemma 5.** *Let  $S = G \wr (B, \bar{B})$  where  $G$  is a finite group and  $\bar{B}$  is the semigroup of constant maps on the set  $B$ . Then  $S$  is simple and the maximal subgroup of  $S$  is isomorphic to  $G$ . More precisely, if  $e = (f, \bar{b})$  is an idempotent then the map  $\psi : eSe \rightarrow G$  given by  $\psi(f', \bar{b}) = bf'$  is an isomorphism.*

*Proof.* We have already observed that  $S$  is simple (this can also be verified by direct computation). Let  $e = (f, \bar{b})$  be an idempotent of  $S$ . We must show  $\psi$  defined as above is an isomorphism. First we verify  $\psi$  is a homomorphism. Indeed,  $(f', \bar{b})(f'', \bar{b}) = (f'\bar{b}f'', \bar{b})$  and  $b(f'\bar{b}f'') = bf'b f''$ . In particular, we have  $1 = \psi(e) = bf$ .

To see  $\psi$  is injective, note that  $(f', \bar{b}) \in G_e$  implies  $(f', \bar{b}) = (f, \bar{b})(f', \bar{b}) = (f\bar{b}f', \bar{b})$  and so  $b'f' = b'fbf'$  all  $b' \in B$ . Thus  $f'$  is determined by  $bf' = \psi(f', \bar{b})$  and so  $\psi$  is injective. Finally to verify  $\psi$  is onto, let  $g \in G$  and consider  $(f', \bar{b}) = e(\bar{g}, \bar{b})e$  where  $\bar{g}$  is the constant map  $B \rightarrow G$  taking all of  $B$  to  $g$ . Then  $bf' = (bf)(b\bar{g})(bf) = g$  since  $bf = 1$ . Thus  $\psi(f', \bar{b}) = g$ , establishing  $\psi$  is onto.  $\square$

### 3. THE PROOFS OF THEOREMS 1 AND 2

In this section we prove Theorems 1 and 2 modulo two technical lemmas. Fix a variety of finite groups  $\mathbf{H}$  closed under extension and containing  $\mathbb{Z}/p\mathbb{Z}$  for infinitely many primes  $p$ . Denote by  $\overline{\mathbf{H}}$  the variety of finite monoids whose subgroups belong to  $\mathbf{H}$ .

*Proof of Theorem 1.* Let  $X = \{x_1, \dots, x_n\}$  be a finite set of cardinality at least two. Denote by  $I$  the minimal ideal of  $\widehat{F}_{\overline{\mathbf{H}}}(X)$ . Choose an idempotent  $e \in I$ . Recall that if  $x$  is an element of a profinite semigroup, then  $x^\omega = \lim x^{n!}$  is the unique idempotent in the closed subsemigroup generated by  $x$ . Without loss of generality we may assume  $x_1^\omega e = e = ex_2^\omega$ ; if not replace  $e$  with  $(x_1^\omega ex_2^\omega)^\omega$ . Let  $G_e$  be the maximal subgroup at  $e$ . Our goal is to show  $G_e$  is free pro- $\mathbf{H}$  on a countable set of generators converging to the identity (that is, free of countable rank). Recall that a subset  $Y$  of a profinite group  $G$  converges to the identity if each neighbourhood of the identity contains all but finitely many elements of  $Y$ . A pro- $\mathbf{H}$  group  $F$  is free pro- $\mathbf{H}$  on a subset  $Y$  converging to the identity if given any map  $\tau : Y \rightarrow H$  with  $H$  pro- $\mathbf{H}$  and  $\tau(Y)$  converging to the identity, there is a unique extension of  $\tau$  to  $F$ . Any free pro- $\mathbf{H}$  group on a profinite space has a basis converging to the identity. See [13] for details.

It is well known  $\widehat{F}_{\overline{\mathbf{H}}}(X)$  is metrizable [2, 12], and hence so is  $G_e$ . Thus the identity  $e$  of  $G_e$  has a countable basis of neighbourhoods. We shall use a well-known criterion, going back to Iwasawa [8], to establish  $G_e$  is free pro- $\mathbf{H}$  of countable rank. An *embedding problem* for  $G_e$  is a diagram

$$\begin{array}{ccc} & G_e & \\ & \downarrow \varphi & \\ H & \xrightarrow{\alpha} & K \end{array} \quad (3.1)$$

with  $H \in \mathbf{H}$  and  $\varphi, \alpha$  epimorphisms ( $\varphi$  continuous). A *solution* to the embedding problem (3.1) is a continuous **epimorphism**  $\tilde{\varphi} : G_e \rightarrow H$  making the diagram

$$\begin{array}{ccc} & G_e & \\ \tilde{\varphi} \swarrow & \downarrow \varphi & \\ H & \xrightarrow{\alpha} & K \end{array}$$

commute. According to [13, Corollary 3.5.10] to prove  $G_e$  is free pro- $\mathbf{H}$  of countable rank it suffices to show that every embedding problem (3.1) for

$G_e$  has a solution. We proceed via a series of reductions on the types of embedding problems we need to consider.

Since  $G_e$  is a closed subgroup of  $\widehat{F}_{\overline{\mathbf{H}}}(X)$ , there is a continuous onto homomorphism  $\varphi' : \widehat{F}_{\overline{\mathbf{H}}}(X) \twoheadrightarrow M'$  with  $M'$  a finite monoid in  $\overline{\mathbf{H}}$  such that  $\ker \varphi'|_{G_e} \leq \ker \varphi$ . Setting  $K' = \varphi'(G_e)$ , let  $\rho : K' \twoheadrightarrow K$  be the canonical projection. Defining  $H'$  to be the pullback of  $\alpha$  and  $\rho$ , that is  $H' = \{(h, k') \in H \times K' \mid \alpha(h) = \rho(k')\}$ , yields a commutative diagram

$$\begin{array}{ccc} & & G_e \\ & \swarrow \varphi' & \searrow \varphi \\ H' & \xrightarrow{\alpha'} & K' \\ \downarrow \rho^* & & \downarrow \rho \\ H & \xrightarrow{\alpha} & K \end{array}$$

where  $\rho^*$  is the projection to  $H$ . It is easily verified that all the arrows in the diagram are epimorphisms. So to solve our original embedding problem, it suffices to solve the embedding problem:

$$\begin{array}{ccc} & G_e & \\ & \downarrow \varphi' & \\ H' & \xrightarrow{\alpha'} & K' \end{array}$$

In other words, reverting back to our original notation, we may assume in the embedding problem (3.1) the map  $\varphi$  is the restriction of a continuous onto homomorphism  $\varphi : \widehat{F}_{\overline{\mathbf{H}}}(X) \twoheadrightarrow M$  with  $M \in \overline{\mathbf{H}}$ . Let  $J$  be the minimal ideal of  $M$ ; so  $J = \varphi(I)$ . Then the right Schützenberger representation [6, 12, 14] of  $M$  on  $J$  is faithful when restricted to  $K$ . Possibly replacing  $M$  by its image under the Schützenberger representation, we may assume that the right Schützenberger representation of  $M$  on  $J$  is faithful. Therefore, we may view  $M$  as embedded in the wreath product  $K \wr (B, \text{RLM}_J(M))$ . The existence of a solution then follows from the following technical lemma, which is the subject of Section 5.

**Lemma 6.** *Let  $\varphi : \widehat{F}_{\overline{\mathbf{H}}}(X) \twoheadrightarrow M$  be a continuous surjective morphism, with  $M$  finite, such that  $\varphi(G_e) = K$  and the (right) Schützenberger representation of  $M$  on its minimal ideal  $J$  is faithful. Let  $\alpha : H \twoheadrightarrow K$  be an epimorphism. Then there is an  $X$ -generated finite monoid  $M' \in \overline{\mathbf{H}}$  such that if  $\eta : \widehat{F}_{\overline{\mathbf{H}}}(X) \rightarrow M'$  is the continuous projection, then:*

- (1) *there is an isomorphism  $\theta : G_{\eta(e)} \rightarrow H$  where  $G_{\eta(e)}$  is the maximal subgroup at  $\eta(e)$  of the minimal ideal of  $M'$ ;*
- (2)  *$\varphi$  factors through  $\eta$  as  $\rho\eta$  where  $\rho : M' \twoheadrightarrow M$  satisfies  $\rho\theta^{-1} = \alpha$ .*

Assuming the lemma, our desired solution to the embedding problem (3.1) is  $\tilde{\varphi} = \theta\eta|_{G_e} : G_e \rightarrow H$ . Indeed,  $\eta|_{G_e}$  is an epimorphism by Proposition 4 and hence  $\tilde{\varphi}$  is an epimorphism. Moreover,  $\alpha\tilde{\varphi} = \rho\theta^{-1}\theta\eta|_{G_e} = \varphi|_{G_e}$  and so

$\tilde{\varphi}$  is indeed a solution to the embedding problem (3.1). This completes the proof of Theorem 1.  $\square$

*Proof of Theorem 2.* Let  $\pi : \widehat{F}_{\mathbf{H}}(X) \rightarrow \widehat{F}_{\mathbf{H}}(X)$  be the canonical projection; so  $\varphi = \pi|_G$  where  $G$  is the maximal subgroup of the minimal ideal  $I$  of  $\widehat{F}_{\mathbf{H}}(X)$ . Let  $N = \ker \varphi$ . We shall use a criterion due to Mel'nikov to prove that  $N$  is free pro- $\mathbf{H}$ . We first need to recall the notion of  $S$ -rank [13]. If  $S$  is a finite simple group and  $G$  is a profinite group, denote by  $M_S(G)$  the intersection of all open normal subgroups  $N$  of  $G$  such that  $G/N \cong S$ . It is known [13, Chapter 8.2] that  $G/M_S(G) \cong \prod_A S$ , a direct product of copies of  $S$  indexed by  $A$ . The cardinality  $r_S(G)$  of  $A$  is called the  $S$ -rank of  $G$ . One property of  $S$ -rank that we shall need is part of [13, Lemma 8.2.5].

**Lemma 7.** *Suppose  $H$  is a continuous image of  $G$ , then  $r_S(H) \leq r_S(G)$ .*

Mel'nikov's criterion for freeness of a normal subgroup [13, Theorem 8.6.8] is then:

**Theorem 8** (Mel'nikov). *Let  $\mathbf{H}$  be a variety of finite groups closed under extension and let  $F$  be a free pro- $\mathbf{H}$  group of countably infinite rank. A non-trivial closed normal subgroup  $N$  of infinite index in  $F$  is free pro- $\mathbf{H}$  (of countable rank) if and only if the  $S$ -rank  $r_S(N)$  is infinite for each finite simple group  $S \in \mathbf{H}$ .*

In our context, since  $G/N$  is a free profinite group of rank  $|X|$ , clearly  $N$  has infinite index. So it suffices to show that  $N$  has infinite  $S$ -rank for all finite simple groups  $S \in \mathbf{H}$ . By Lemma 7 it suffices to show  $S^n$  is a continuous image of  $N$  for all  $n \geq 1$  (as  $r_S(S^n) = n$ ). Notice that  $\pi(E(I)) = 1$  and so  $\langle E(I) \rangle \cap G_e \leq N$  (one can in fact show that  $N$  is the closed normal subgroup generated by  $\langle E(I) \rangle \cap G_e$ , but we shall not need this). The desired result is then an immediate consequence of the following technical lemma, which will be proved in Section 4.

**Lemma 9.** *Let  $\mathbf{H}$  be any variety of finite groups containing cyclic groups of arbitrary cardinality and let  $H \in \mathbf{H}$ . Then there is a two-generated finite monoid  $M \in \overline{\mathbf{H}}$  such that  $H$  is the maximal subgroup of the minimal ideal  $J$  of  $M$  and  $J$  is generated by idempotents.*

From Lemma 9 we conclude every group in  $\mathbf{H}$  is a continuous image of  $N$ , yielding Theorem 2. Indeed, if  $\psi : \widehat{F}_{\mathbf{H}}(X) \rightarrow M$  is the canonical surjection, then  $\psi(E(I)) = E(\psi(I))$  and so, since  $M$  is finite,

$$\psi(\overline{\langle E(I) \rangle}) = \psi(\langle E(I) \rangle) = \langle \psi(E(I)) \rangle = \langle E(J) \rangle = J$$

as  $J = \psi(I)$ . By Graham's theorem [7, 12] the idempotent-generated sub-semigroup of a finite simple semigroup is simple; hence the closed sub-semigroup generated by the idempotents of a simple profinite semigroup is simple. Proposition 4 then easily yields  $\psi(N)$  is the maximal subgroup of  $J$ .

## 4. THE PROOF OF LEMMA 9

We prove Lemma 9 first since the proof is easier and at the same time highlights many of the ideas that will be used to prove Lemma 6. The construction we use is a variant on a classical construction. Usually it is formulated in terms of Rees matrix semigroups, but it will be more convenient for us to use wreath products. If  $(Y, S)$  is a transformation monoid, then  $\overline{(Y, S)}$  denotes the augmented transformation monoid obtained by adjoining to  $S$  the constant maps on  $Y$ . Set  $[n] = \{1, \dots, n\}$ .

Let  $H \in \mathbf{H}$  and let  $n \geq 2|H| - 1$  be an integer such that  $\mathbb{Z}/n\mathbb{Z}$  belongs to  $\mathbf{H}$ . Suppose  $e$  is the identity of  $H$  and  $H = \{e = h_1, \dots, h_m\}$ . Let  $C_n$  be the cyclic group of order  $n$  generated by the cyclic permutation  $a = (1\ 2 \cdots n)$ . Consider the following two elements of the wreath product  $H \wr (\overline{[n]}, C_n)$ :

$$x = (\overline{e}, a) \text{ where } j\overline{e} = e, \text{ all } j \in [n]$$

$$y = (Y, \overline{1}) \text{ where } jY = \begin{cases} h_j & 1 \leq j \leq m \\ e & m < j \leq n. \end{cases}$$

Let  $M$  be the submonoid generated by  $x$  and  $y$ . First observe  $x$  is an invertible element of order  $n$ . On the other hand  $y$  is an idempotent since  $y^2 = (Y\overline{1}Y, \overline{1})$  and  $j(Y\overline{1}Y) = (jY)(1Y) = (jY)h_1 = (jY)e = jY$ . A routine application of Lemma 5 yields the minimal ideal  $J$  of  $M$  is  $M \cap H \wr ([n], \overline{[n]})$ . Indeed,  $M \cap H \wr ([n], \overline{[n]})$  contains  $y$  and is a subsemigroup of the simple semigroup  $H \wr ([n], \overline{[n]})$  (Lemma 5) and hence is simple. Consideration of the projection  $M \rightarrow (\overline{[n]}, C_n)$ , which is onto by definition of  $x, y$ , shows  $J \subseteq M \cap H \wr ([n], \overline{[n]})$ , since  $\overline{[n]}$  is the minimal ideal of  $(\overline{[n]}, C_n)$ . We conclude  $J = M \cap H \wr ([n], \overline{[n]})$  and in particular  $y \in J$ .

According to Lemma 5 the map sending  $(f, \overline{1})$  to  $1f$  restricts to an isomorphism  $\theta$  from the maximal subgroup of  $H \wr ([n], \overline{[n]})$  at  $y$  to  $H$ . We show that  $\theta$  is still surjective when restricted to the maximal subgroup  $G_y$  of  $M$  at  $y$ . In fact, we show that each element of  $H$  is  $\theta(z)$  for some  $z \in G_y$  which is a product of idempotents. Graham's theorem [7, 12] implies a simple semigroup is generated by idempotents if and only if each element of the maximal subgroup is a product of idempotents and so this will complete the proof of Lemma 9. Actually, the proof of Theorem 2 can be made to work using only that each element of  $G_y$  is a product of idempotents.

The key observation is  $x^j y x^{-j}$  is an idempotent for  $1 \leq j \leq m - 1$  and

$$x^j y x^{-j} = (\overline{e}, a^j)(Y, \overline{1})(\overline{e}, a^{-j}) = (a^j Y, \overline{n - j + 1}).$$

Therefore, the element  $z \in G_y$  defined by

$$z = y x^j y x^{-j} = (Y, \overline{1})(a^j Y, \overline{n - j + 1})(Y, \overline{1}) = (Y(\overline{j+1}Y), \overline{n-j+1}Y), \overline{1})$$

is a product of idempotents and

$$\theta(z) = 1 \left( Y(\overline{j+1}Y)(\overline{n-j+1}Y) \right) = 1Y \cdot (j+1)Y \cdot (n-j+1)Y = h_{j+1}$$

since  $1Y = e = (n-j+1)Y$ , where the latter equality holds since  $n \geq 2m-1$  implies  $n-j+1 \geq n-m+2 \geq m+1$ . This establishes each element of  $H$  is  $\theta(z)$  for some  $z \in G_y$  which is a product of idempotents, finishing the proof of Lemma 9.  $\square$

## 5. THE PROOF OF LEMMA 6

The proof of Lemma 6 relies heavily on the wreath product and forms the technical core of this paper. The construction is reminiscent of the one used in the proof of Lemma 9. We shall find it convenient to use the formulation of wreath products in terms of row monomial matrices. Let  $S$  be a semigroup. Then  $RM_n(S)$  denotes the monoid of all  $n \times n$  row monomial matrices with entries in  $S$ ; in other words it consists of all matrices over  $S \cup \{0\}$  such that each row has exactly one non-zero entry. The binary operation is usual matrix multiplication. It is well known [9, 12]  $RM_n(S) \cong S \wr ([n], T_n)$  where  $T_n$  is the full transformation monoid of degree  $n$ . An element  $(f, a)$  corresponds to the matrix  $M$  with  $M_{i,ia} = if$ ,  $1 \leq i \leq n$ , and all other entries zero. In particular, if  $a$  is a constant map to  $j$ , then  $M$  has all its non-zero entries in column  $j$ .

From this viewpoint, an iterated wreath product  $S \wr (B, T) \wr (A, U)$  can be viewed as  $|A| \times |A|$  block row monomial matrices where the blocks are  $|B| \times |B|$  row monomial matrices over  $S$ . The term *block entry* shall mean a matrix from  $S \wr (B, T)$  while the term *entry* shall always mean an element of the semigroup  $S$ . Having dispensed with the preliminaries, we now turn to the proof of Lemma 6

Let  $B$  be the set of  $\mathbf{L}$ -classes of  $J$ . Denote by  $\text{RLM}_J(M)$  the quotient of  $M$  by the kernel of its action on the right of  $B$ ; note that  $\text{RLM}_J(M)$  contains all the constant maps. Since the Schützenberger representation of  $M$  on  $I$  is faithful, we can view  $M$  as a monoid of  $b \times b$  row monomial matrices over  $K$  where  $b = |B|$ . Moreover, the discussion in Section 2 shows that an element  $k$  of the maximal subgroup  $K$  at  $\varphi(e)$  can be identified with the row monomial matrix having  $k$  in every entry of the first column. For  $x \in \widehat{F}_{\mathbf{H}}(X)$ , denote by  $M_x$  the row monomial matrix associated to  $\varphi(x)$ .

Let  $N = \ker \alpha$  and choose a set-theoretic section  $\sigma : K \rightarrow H$ . Then  $H = N\sigma(K)$ . Denote by  $M_x^\sigma$  the row monomial matrix over  $H$  obtained from  $M_x$  by applying  $\sigma$  entry-wise. Let  $n = |N|$  and let  $m$  be a positive integer such that  $(M_{x_1}^\sigma)^m$  is idempotent. Choose a prime  $p > \max\{m, n^b\}$  so that  $\mathbb{Z}/p\mathbb{Z} \in \mathbf{H}$ ; such a prime exists by our assumption on  $\mathbf{H}$ . Denote by  $C_p$  the cyclic group of order  $p$  generated by the permutation  $(1\ 2 \cdots p)$ . Our monoid  $M'$  will be a certain submonoid of the iterated wreath product

$$W = H \wr (B, \text{RLM}_J(M)) \wr (\overline{[p]}, C_p).$$

Observe that  $W \in \overline{\mathbf{H}}$ .



We begin our construction of  $M'$  by defining

$$\tilde{x}_1 = \begin{bmatrix} 0 & M_{x_1}^\sigma & 0 & \cdots & 0 \\ 0 & 0 & M_{x_1}^\sigma & 0 & \cdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & M_{x_1}^\sigma \\ M_{x_1}^\sigma & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

In other words  $\tilde{x}_1$  acts on the  $[p]$  component by the cyclic permutation  $(1\ 2\ \cdots\ p)$  and each block entry of  $\tilde{x}_1$  from  $H \wr (B, \text{RLM}_J(M))$  is  $M_{x_1}^\sigma$ . Set  $\ell = n^b$ ; so  $p > \ell$  by choice of  $p$ . Let  $1 = N_1, N_2, \dots, N_\ell$  be the distinct elements of  $N^b$ . We identify  $N^b$  with the group of diagonal  $b \times b$  matrices over  $N$ . In particular,  $N^b$  is a subgroup of  $H \wr (B, \text{RLM}_J(M))$ , as  $M$  is a monoid. In fact, there is a natural onto homomorphism

$$\bar{\alpha} : H \wr (B, \text{RLM}_J(M)) \rightarrow K \wr (B, \text{RLM}_J(M))$$

induced by  $\alpha : H \rightarrow K$  and, moreover, it is straightforward to verify that  $\bar{\alpha}(x) = \bar{\alpha}(y)$  if and only if  $x = N_j y$  some  $1 \leq j \leq \ell$ . The map  $\bar{\alpha}$  simply applies  $\alpha$  entry-wise.

Next let us define, for  $i = 2, \dots, n$ , a  $p \times p$  block row monomial matrix by

$$\tilde{x}_i = \begin{bmatrix} M_{x_i}^\sigma & 0 & \cdots & 0 \\ N_2 M_{x_i}^\sigma & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ N_\ell M_{x_i}^\sigma & 0 & \cdots & 0 \\ M_{x_i}^\sigma & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ M_{x_i}^\sigma & 0 & \cdots & 0 \end{bmatrix}$$

so  $\tilde{x}_i$  has all its non-zero blocks in the first column. The  $j^{\text{th}}$  block entry of the first column is  $N_j M_{x_i}^\sigma$  if  $j \leq \ell$  and otherwise is  $M_{x_i}^\sigma$ . Then  $\tilde{x}_1, \dots, \tilde{x}_n \in W$  and we have a map  $X \rightarrow W$  given by  $x_i \mapsto \tilde{x}_i$ . Extend this to a continuous morphism  $\eta : \widehat{F}_{\mathbf{H}}(X) \rightarrow W$  and set  $M' = \eta(\widehat{F}_{\mathbf{H}}(X))$ . Our goal is to show  $M'$  is the desired monoid. We begin by verifying that  $\varphi$  factors through  $\eta$ .

**Proposition 10.** *Let  $u \in \widehat{F}_{\mathbf{H}}(X)$ . Then each  $U \in H \wr (B, \text{RLM}_J(M))$  appearing as a block entry of  $\eta(u)$  satisfies  $\bar{\alpha}(U) = M_u$ . As a consequence  $\eta(u) = \eta(u')$  implies  $\varphi(u) = \varphi(u')$  and so  $\varphi$  factors through  $\eta$  as  $\rho\eta$  where  $\rho : M \rightarrow M'$  takes  $\eta(u)$  to  $\bar{\alpha}(U)$  where  $U$  is any block entry of  $\eta(u)$ .*

*Proof.* The second statement is immediate from the first. We prove the first statement for words  $w \in X^*$  by a simple induction on length, the case  $|w| = 0$  being trivial. If  $w = x_1 u$ , then the definition of  $\tilde{x}_1$  implies the block entries of  $\eta(w)$  are of the form  $M_{x_1}^\sigma U$  where  $U$  runs over the block entries of  $\eta(u)$  and the result follows by induction. The case  $w = x_i u$ ,  $2 \leq i \leq n$ , is similar only the block entries of  $\eta(w)$  are now of the form  $N_j M_{x_i}^\sigma U$  with  $U$  the block entry of  $\eta(u)$  in the first row. If  $u \in \widehat{F}_{\mathbf{H}}(X)$ , then since  $X^*$  is

dense, there exists a word  $w \in X^*$  such that  $\eta(u) = \eta(w)$  and  $M_u = M_w$ . The result now follows from the case of words.  $\square$

Our next goal is to show that if  $w$  is a word whose support contains some letter other than  $x_1$ , then each preimage of  $M_w$  under  $\bar{\alpha}$  is a block entry of  $\eta(w)$ . This will be crucial in showing that the maximal subgroup of the minimal ideal of  $M'$  is isomorphic to  $H$ . To effect this we shall need the following lemma.

**Lemma 11.** *Let  $u, w \in \widehat{F}_{\bar{H}}(X)$  and suppose  $W_1, \dots, W_\ell$  are the preimages of  $M_w$  under  $\bar{\alpha}$  and  $U$  is a fixed preimage of  $M_u$  under  $\bar{\alpha}$ . Then  $W_1U, \dots, W_\ell U$ , respectively  $UW_1, \dots, UW_\ell$ , are all the preimages of  $M_w M_u$ , respectively  $M_u M_w$ , under  $\bar{\alpha}$ .*

*Proof.* The preimages of  $M_w$  under  $\bar{\alpha}$  are  $N_1 M_w^\sigma, \dots, N_\ell M_w^\sigma$ . But as  $M_w^\sigma U$  is a preimage of  $M_w M_u$ , it follows  $\{N_1 M_w^\sigma U, \dots, N_\ell M_w^\sigma U\}$  is the complete set of preimages of  $M_w M_u$  under  $\bar{\alpha}$ . For the preimages of  $M_u M_w$ , note that  $UN_1, \dots, UN_\ell$  are the  $\bar{\alpha}$ -preimages of  $M_u$  so the previous case applies.  $\square$

Observe that if  $w \in X^*$  and the support of  $w$  is not contained in  $\{x_1\}$ , then by definition of  $\tilde{x}_2, \dots, \tilde{x}_n$ , the block entries of  $\eta(w)$  form a single column, that is the  $([p], C_p)$  component of  $\eta(w)$  is a constant map. We can now prove the aforementioned fact concerning preimages.

**Proposition 12.** *Let  $w \in X^*$  have support not contained in  $\{x_1\}$ . Then each preimage of  $M_w$  under  $\bar{\alpha}$  appears as a block entry of  $\eta(w)$ .*

*Proof.* Let  $S$  be the set of words in  $X^*$  with support containing an element outside of  $\{x_1\}$ . We proceed by induction on  $|w|$ . If  $|w| = 1$ , then the proposition follows from the definition of  $\tilde{x}_2, \dots, \tilde{x}_n$ .

Suppose it is true for words in  $S$  of length  $n$  and let  $w \in S$  have length  $n + 1$ . If the first letter of  $w \neq x_1$ , then  $w = ux_i$  with  $u \in S$  some  $i$ ; else  $w = x_1 u$  where  $u \in S$ . In the case  $w = x_1 u$  the block entries of  $\eta(w)$  are precisely the products of the form  $M_{x_1}^\sigma U$  where  $U$  runs over the block entries of  $\eta(u)$ . By induction and Lemma 11 it follows that the block entries of  $\eta(w)$  are as required. In the case  $w = ux_i$ , the block entries of  $\eta(u)$  are in a single column, say column  $j$ . Let  $V$  be the block entry in row  $j$  of  $\tilde{x}_i$ . Then the block entries of  $\eta(w)$  are all products of the form  $UV$  where  $U$  is a block entry of  $\eta(u)$ . So again Lemma 11 yields each  $\bar{\alpha}$ -preimage of  $M_w$  is a block entry of  $\eta(w)$ . This completes the proof.  $\square$

**Corollary 13.** *If  $w \in I$ , then the block entries of  $\eta(w)$  are in a single column and each preimage under  $\bar{\alpha}$  of  $M_w$  appears as a block entry of  $\eta(w)$ .*

*Proof.* Since  $\bar{H}$  contains the free semilattice  $(P(X), \cup)$  it follows that if  $\{w_r\}$  is a sequence of words in  $X^*$  converging to  $w$ , then there exists  $R$  such that for  $r \geq R$  the word  $w_r$  has support  $X$ . Now there exists  $s \geq R$  so that  $\eta(w) = \eta(w_s)$ . Since  $w_s$  has full support, the corollary then follows from Proposition 12 and the remark preceding that proposition.  $\square$

By Corollary 13 if  $w \in I$ , then the  $\overline{([p], C_p)}$  component of  $\eta(w)$  is a constant map, that is the block entries of  $\eta(w)$  appear in a single column. Moreover, Proposition 10 shows that each block entry of  $\eta(w)$  is a preimage of  $M_w$  under  $\bar{\alpha}$ . But  $M_w$ , being in the minimal ideal, has the shape of a constant map, i.e. it has only one non-zero column. Hence  $\eta(w)$  has all its entries in a single column, that is, the  $(B, \text{RLM}_J(M)) \wr \overline{([p], C_p)}$  component of  $\eta(w)$  is a constant map. Since  $\eta(I)$  is the minimal ideal  $J'$  of  $M'$  (Proposition 4), we conclude  $J' \subseteq M' \cap H \wr (B \times [p], \overline{B \times [p]})$  and hence is simple by Lemma 5. Since simple semigroups form a variety of finite semigroups,  $M' \cap H \wr (B \times [p], \overline{B \times [p]})$  is simple. Therefore, we in fact have  $J' = M' \cap H \wr (B \times [p], \overline{B \times [p]})$ . It remains to construct an isomorphism  $\theta : G_{\eta(e)} \rightarrow H$  such that  $\rho\theta^{-1} = \alpha$ .

First note that since  $ex_2^\omega = e$ , it must be the case  $\eta(e)$  is a block matrix with each block entry in the first column. Also, the discussion in Section 2 indicates  $M_e$  is a matrix whose only non-zero column is the first column and whose non-zero entries are comprised by the identity of  $K$ . Since the block entries of  $\eta(e)$  are preimages of  $M_e$  under  $\bar{\alpha}$  (Proposition 10), we deduce that all the entries of  $\eta(e)$  are in the first column and belong to  $N$ . Lemma 5 says the map  $\theta : H \wr (B \times [p], \overline{B \times [p]}) \rightarrow H$  selecting the 1,1 entry is an isomorphism from the maximal subgroup at  $\eta(e)$  of  $H \wr (B \times [p], \overline{B \times [p]})$  to  $H$ . In particular, the 1,1 entry of  $\eta(e)$  is the identity of  $H$ .

We need to show that the restriction of  $\theta$  to  $G_{\eta(e)}$  is onto and  $\rho\theta^{-1} = \alpha$ . Let us prove the second assertion assuming the first. By Proposition 10 if  $w \in \widehat{F}_{\overline{\mathbf{H}}}(X)$  maps under  $\eta$  to  $\theta^{-1}(h)$ , then  $\varphi(w) = \rho\eta(w)$  is obtained by choosing say the 1,1 block entry of  $\eta(w)$  and applying  $\alpha$  entry-wise. Since an element  $k$  of  $K = G_{\varphi(e)}$ , viewed as a row monomial matrix, has  $k$  as each non-zero entry, it follows the image of  $\theta^{-1}(h)$  in  $K$  is obtained by evaluating  $\alpha$  on the 1,1 entry of  $\eta(w)$ . But the 1,1 entry of  $\eta(w)$  is precisely  $h$  by definition of  $\theta$  so  $\rho\theta^{-1}(h) = \alpha(h)$ , as required.

Thus we are left with proving  $\theta$  is onto. Since  $\rho$  must take  $G_{\eta(e)}$  onto  $K$  (Proposition 4), it follows from the discussion in the previous paragraph that we must be able to obtain a preimage under  $\alpha$  of each element of  $K$  as the 1,1 entry of some element of  $G_{\eta(e)}$ , that is,  $\alpha(\theta(G_{\eta(e)})) = K$ . So it suffices to prove  $\ker \alpha = N$  is contained in the image of  $\theta$ .

Recall that  $p$  was chosen so that  $p > m$  where  $(M_{x_1}^\sigma)^m = (M_{x_1}^\sigma)^\omega$ . We can thus find a positive integer  $r$  so that  $1 \equiv rm \pmod{p}$ . Then

$$\tilde{x}_1^{mr} = \begin{bmatrix} 0 & (M_{x_1}^\sigma)^\omega & 0 & \cdots & 0 \\ 0 & 0 & (M_{x_1}^\sigma)^\omega & 0 & \cdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & (M_{x_1}^\sigma)^\omega \\ (M_{x_1}^\sigma)^\omega & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Set  $C = \tilde{x}_1^{mr}$ . Then  $C^j$  has the block form of the permutation matrix corresponding to  $(1 \ 2 \ \cdots p)^j$  and each block entry of  $C^j$  is  $(M_{x_1}^\sigma)^\omega$ . By

Corollary 13 each preimage of  $M_e$  under  $\bar{\alpha}$  appears as a block entry of  $\eta(e)$ . Since  $\bar{\alpha}((M_{x_1}^\sigma)^\omega) = M_{x_1}^\omega$  and  $x_1^\omega e = e$ , it follows from Lemma 11 that the elements of the form  $(M_{x_1}^\sigma)^\omega U$ , where  $U$  runs over the block entries of  $\eta(e)$ , yield all the preimages of  $M_e$  under  $\bar{\alpha}$  (with perhaps some repetition). Each such matrix is the 1, 1 block entry of a product  $C^j M_e$  for a correctly chosen  $j$ . Now the  $\bar{\alpha}$ -preimages of  $M_e$  are the matrices whose first column has entries from  $N$  and whose remaining columns consist of zeroes. Consequently any element of  $N$  can be the 1, 1 entry of an  $\bar{\alpha}$ -preimage of  $M_e$  and so every element of  $N$  is the 1, 1 entry of some  $C^j \eta(e)$ . Since  $\eta(e)$  has the identity of  $H$  in the 1, 1 entry,  $\eta(e)C^j \eta(e)$  is an element of  $G_{\eta(e)}$  with the same 1, 1 entry as  $C^j \eta(e)$ . Thus  $\theta(G_{\eta(e)})$  contains  $N$  as required. This completes the proof of Lemma 6, thereby establishing Theorems 1 and 2.  $\square$

## REFERENCES

- [1] D. Allen, Jr. and J. Rhodes. Synthesis of the classical and modern theory of finite semigroups. *Advances in Math.*, 11(2):238–266, 1973.
- [2] J. Almeida. *Finite semigroups and universal algebra*, volume 3 of *Series in Algebra*. World Scientific Publishing Co. Inc., River Edge, NJ, 1994.
- [3] J. Almeida. Profinite groups associated with weakly primitive substitutions. *Fundam. Prikl. Mat.*, 11(3):13–48, 2005.
- [4] J. Almeida and M. V. Volkov. Profinite identities for finite semigroups whose subgroups belong to a given pseudovariety. *J. Algebra Appl.*, 2(2):137–163, 2003.
- [5] J. Almeida and M. V. Volkov. Subword complexity of profinite words and subgroups of free profinite semigroups. *Internat. J. Algebra Comput.*, 16(2):221–258, 2006.
- [6] A. H. Clifford and G. B. Preston. *The algebraic theory of semigroups. Vol. I*. Mathematical Surveys, No. 7. American Mathematical Society, Providence, R.I., 1961.
- [7] R. L. Graham. On finite 0-simple semigroups and graph theory. *Math. Systems Theory*, 2:325–339, 1968.
- [8] K. Iwasawa. On solvable extensions of algebraic number fields. *Ann. of Math (2)*, 58:548–572, 1953.
- [9] K. Krohn, J. Rhodes, and B. Tilson. *Algebraic theory of machines, languages, and semigroups*. Edited by Michael A. Arbib. Academic Press, New York, 1968. Chapters 1, 5–9.
- [10] J. Rhodes and B. Steinberg. Profinite semigroups, varieties, expansions and the structure of relatively free profinite semigroups. *Internat. J. Algebra Comput.*, 11(6):627–672, 2001.
- [11] J. Rhodes and B. Steinberg. Closed subgroups of free profinite monoids are projective profinite groups. *Bull. London Math. Soc.*, to appear.
- [12] J. Rhodes and B. Steinberg. *The q-theory of finite semigroups*. Springer, To appear.
- [13] L. Ribes and P. Zalesskii. *Profinite groups*, volume 40 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A*. Springer-Verlag, Berlin, 2000.
- [14] M.-P. Schützenberger. Sur la représentation monomiale des demi-groupes. *C. R. Acad. Sci. Paris*, 246:865–867, 1958.

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